Discrepancy of generalized Hammersley type point sets in Besov spaces with dominating mixed smoothness

Lev Markhasin*

Friedrich-Schiller-Universität Jena, e-mail: lev.markhasin@uni-jena.de

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The symmetrized Hammersley point set is known to achieve the best possible rate for the L_2-norm of the discrepancy function. Also lower bounds for the norm in Besov spaces with dominating mixed smoothness are known. In this paper a large class of point sets which are generalizations of the Hammersley type point sets are proved to asymptotically achieve the known lower bound of the Besov norm. The proof uses a b-adic generalization of the Haar system. This result can be regarded as a preparation for the proof in arbitrary dimension.

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1 Introduction

Let N be some positive integer and $\mathcal{P}$ a point set in the unit cube $\mathbb{Q}^d = [0,1)^d$ with N points. Then the discrepancy function $D_{\mathcal{P}}$ is defined as

$$D_{\mathcal{P}}(x) = \frac{1}{N} \sum_{z \in \mathcal{P}} \mathbb{1}_{C_z}(x) - |B_x|. \tag{1}$$

By $|B_x| = x_1 \cdot \ldots \cdot x_d$ we denote the volume of the rectangular box $B_x = [0, x_1) \times \ldots \times [0, x_d)$ where $x = (x_1, \ldots, x_d) \in \mathbb{Q}^d$ while $\mathbb{1}_{C_z}$ is the characteristic function of the rectangular box $C_z = (z_1, 1) \times \ldots \times (z_d, 1)$ for $z \in \mathcal{P}$. So the discrepancy function measures the deviation between the number of points of $\mathcal{P}$ in B_x and the fair number of points $N|B_x|$ which would be achieved by a (practically impossible) perfect uniform distribution of the points of $\mathcal{P}$, normalized by the total number of points.

Usually one is interested in calculating the norm of the discrepancy function in some normed space of functions on $\mathbb{Q}^d$ to which the discrepancy function belongs. A very well known result refers to the space $L_2(\mathbb{Q}^d)$. It was proved by Roth in [R54]. There exists a constant $c_1 > 0$ such that for any $N \geq 1$ the discrepancy function of any point set $\mathcal{P}$ in $\mathbb{Q}^d$ with N points satisfies

$$\|D_{\mathcal{P}}|L_2\| \geq c_1 \, \frac{(\log N)^{\frac{d-1}{2}}}{N}.$$

The currently best known values for the constant c_1 can be found in [HM11]. Furthermore, there exists a constant $c_2 > 0$ such that for any $N \geq 1$, there exists a point set $\mathcal{P}$ in $\mathbb{Q}^d$ with N points that satisfies

$$\|D_{\mathcal{P}}|L_2\| \leq c_2 \, \frac{(\log N)^{\frac{d-1}{2}}}{N}.$$

This result is known for dimension 2 from [D56] (Davenport), for dimension 3 from [R79] (Roth) and for arbitrary dimension from [R80] (Roth). Only Davenport's result has been proved by an explicit construction while for higher dimensions probabilistic methods were used until Chen and Skriganov found explicit constructions for arbitrary dimension in [CS02]. Results for the constant c_2 can be found in [FPPS10].

Both bounds were extended to L_p-spaces for any $1 < p < \infty$. In the case of the lower bound the reference is [S77] (Schmidt) while for the upper bound it is [C81]

(Chen).

As general references for studies of the discrepancy function we refer to the recent monographs [DP10] and [NW10] as well as [M99], [KN74] and [B11].

Until recently other norms than L_p-norms weren't studied a lot in the context of discrepancy. Triebel started the study of the discrepancy function in other function spaces like Sobolev, Besov and Triebel-Lizorkin spaces in [T10b] and [T10a]. In [H10] Hinrichs proved sharp upper bounds for the norms in Besov spaces with dominating mixed smoothness. Triebel's result was that for all $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}$ satisfying $\frac{1}{p} - 1 < r < \frac{1}{p}$ and $q < \infty$ if $p = 1$ and $q > 1$ if $p = \infty$ there exist constants $c_1, c_2 > 0$ such that, for any $N \geq 2$, the discrepancy function of any point set $\mathcal{P}$ in $\mathbb{Q}^d$ with N points satisfies

$$\left\| D_{\mathcal{P}} | S^r_{pq} B(\mathbb{Q}^d) \right\| \geq c_1 \, N^{r-1} (\log N)^{\frac{d-1}{q}}, \tag{2}$$

and, for any $N \geq 2$, there exists a point set $\mathcal{P}$ in $\mathbb{Q}^d$ with N points such that

$$\left\| D_{\mathcal{P}} | S^r_{pq} B(\mathbb{Q}^d) \right\| \leq c_2 \, N^{r-1} (\log N)^{(d-1)(\frac{1}{q}+1-r)}.$$

Hinrichs' result closed this gap in the case $d = 2$, we will mention it later.

This note will closely orientate itself on [H10] in terms of structure and methods of proofs. We mention some definitions from [T10a] which are most important for our purpose.

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions on $\mathbb{R}^d$. For $f \in \mathcal{S}'(\mathbb{R}^d)$, we denote by $\mathcal{F}f$ the Fourier transform of f. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\varphi_0(t) = 1$ for $|t| \leq 1$ and $\varphi_0(t) = 0$ for $|t| > \frac{3}{2}$. Let

$$\varphi_k(t) = \varphi_0(2^{-k} t) - \varphi_0(2^{-k+1} t)$$

where $t \in \mathbb{R}$, $l \in \mathbb{N}$ and

$$\varphi_k(t) = \varphi_{k_1}(t_1) \dots \varphi_{k_d}(t_d)$$

where $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d$, $t = (t_1, \dots, t_d) \in \mathbb{R}^d$. The functions φ_k are a dyadic resolution of unity since

$$\sum_{k \in \mathbb{N}_0^d} \varphi_k(x) = 1$$

for all $x \in \mathbb{R}^d$. The functions $\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)$ are entire analytic functions for any $f \in \mathcal{S}'(\mathbb{R}^d)$.

Let $0 < p, q \leq \infty$ and $r \in \mathbb{R}$. The Besov space with dominating mixed smoothness $S_{pq}^r B(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with finite quasi-norm

$$\left\| f | S_{pq}^r B(\mathbb{R}^d) \right\| = \left(\sum_{k \in \mathbb{N}_0^d} 2^{r(k_1 + \ldots + k_d)q} \left\| \mathcal{F}^{-1}(\varphi_k \mathcal{F} f) | L_p(\mathbb{R}^d) \right\|^q \right)^{\frac{1}{q}}$$

with the usual modification if $q = \infty$.

Let $\mathcal{D}(\mathbb{Q}^d)$ consist of all complex-valued infinitely differentiable functions on $\mathbb{R}^d$ with compact support in the interior of $\mathbb{Q}^d$ and let $\mathcal{D}'(\mathbb{Q}^d)$ be its dual space of all distributions in $\mathbb{Q}^d$. The Besov space with dominating mixed smoothness $S_{pq}^r B(\mathbb{Q}^d)$ consists of all $f \in \mathcal{D}'(\mathbb{Q}^d)$ with finite quasi-norm

$$\left\| f | S_{pq}^r B(\mathbb{Q}^d) \right\| = \inf \left\{ \left\| g | S_{pq}^r B(\mathbb{R}^d) \right\| : \, g \in S_{pq}^r B(\mathbb{R}^d), \, g|_{\mathbb{Q}^d} = f \right\}.$$

The spaces $S_{pq}^r B(\mathbb{R}^d)$ and $S_{pq}^r B(\mathbb{Q}^d)$ are quasi-Banach spaces.

In [H10] Hinrichs analyzed the norm of the discrepancy function of point sets of the Hammersley type in Besov spaces with dominating mixed smoothness. He proved upper bounds which are special cases of our results in this note. The result from [H10] is that for $r \geq 0$ there is a constant $c > 0$ such that for any $N \geq 2$, there exists a point set $\mathcal{P}$ in $\mathbb{Q}^2$ with N points such that

$$\left\| D_{\mathcal{P}} | S_{pq}^r B(\mathbb{Q}^2) \right\| \leq c \, N^{r-1} (\log N)^{\frac{1}{q}}.$$

This result closed the gap of Triebel's results in dimension 2. In this note we prove the same bound for a larger class of point sets. Hinrichs used point sets of Hammersley type. We use generalizations of these point sets.

For any integer $b \geq 2$ we consider the following point sets

$$\mathcal{R}_n = \left\{ \left(\frac{t_n}{b} + \frac{t_{n-1}}{b^2} + \ldots + \frac{t_1}{b^n}, \frac{s_1}{b} + \frac{s_2}{b^2} + \ldots + \frac{s_n}{b^n} \right) | \, t_1, \ldots, t_n \in \{0, 1, \ldots, b-1\} \right\}$$

for some $n \in \mathbb{N}$ where for any $i = 1, \ldots, n$ either $s_i = t_i$ or $s_i = b - 1 - t_i$. So, the set $\mathcal{R}_n$ contains b^n points. These sets are called generalized Hammersley type point sets since they generalize original Hammersley type point sets proposed by Hammersley in [H60]. They were first defined by Faure in [F81] and used in [FP09] and [FPPS10] to calculate their L_2-discrepancy.

The explicit constructions for the L_2-discrepancy by Chen and Skriganov from [CS02] use b-adic constructions, similar to the b-adic generalizations of the Ham-

mersley type point sets for $d \geq 2$. One might conjecture that these constructions could be optimal for the norms in Besov spaces with dominating mixed smoothness for arbitrary dimension. Considering this aspect, one could see the current paper as the preparation for the proof of this conjecture.

For any point set $\mathcal{R}_n$ we denote $a_n = \#\{i = 1, \dots, n : \ s_i = t_i\}$. The main result of this note is

Theorem 1.1. *Let $1 \leq p, q \leq \infty$ and $0 \leq r < \frac{1}{p}$. Then for any integer $b \geq 2$ there is a constant $c > 0$ such that for any $n \in \mathbb{N}$ and any generalized Hammersley type point set $\mathcal{R}_n$ with a_n satisfying $|2a_n - n| \leq c_0$ for some constant $c_0 > 0$, we have*

$$\left\| D_{\mathcal{R}_n} | S^r_{pq} B(\mathbb{Q}^2) \right\| \leq c\, b^{n(r-1)}\, n^{\frac{1}{q}}.$$

Remark. *The constant c_0 is independent of n, securing that $|2a_n - n|$ can be estimated with the same constant for any n and any possible $\mathcal{R}_n$. In [H10] only point sets with $a_n = \lfloor \frac{n}{2} \rfloor$ were used (with $b = 2$). So a possible value for c_0 in that case would be* 1.

In order to prove the result we will calculate b-adic Haar coefficients of the discrepancy function.

The distribution of points in a cube is not just a theoretical concept. Its application in quasi-Monte Carlo methods is very important. Quadrature formulas need very well distributed point sets. The connection of discrepancy and the error of quadrature formulas can be given for a lot of norms. In [T10a, Theorem 6.11] Triebel gave this connection for Besov spaces with dominating mixed smoothness. We define the error of the quadrature formulas in some Banach space $M(\mathbb{Q}^d)$ of functions on $\mathbb{Q}^d$ with N points as

$$\mathrm{Err}_N(M(\mathbb{Q}^d)) = \inf_{\{x_1,\dots,x_N\} \subset \mathbb{Q}^d} \sup_{f \in M^1_0(\mathbb{Q}^d)} \left| \int_{\mathbb{Q}^d} f(x)\mathrm{d}x - \frac{1}{N} \sum_{k=1}^N f(x_k) \right|$$

where by $M^1_0(\mathbb{Q}^d)$ we mean the subset of the unit ball of $M(\mathbb{Q}^d)$ with the property that for all $f \in M^1_0(\mathbb{Q}^d)$ its extension to $\overline{\mathbb{Q}^d}$ vanishes whenever one of the coordinates of the argument is 1.

Theorem 1.2. *Let $1 \leq p, q \leq \infty$ and $0 \leq r < \frac{1}{p}$. Then for any integer $b \geq 2$ there are constants $c_1, c_2 > 0$ such that, for any $n \in \mathbb{N}$ and any generalized Hammersley type point set $\mathcal{R}_n$ with a_n satisfying $|2a_n - n| \leq c_0$ for some constant $c_0 > 0$, we have*

$$c_1 \frac{(\log)^{\frac{q(d-1)}{q-1}}}{N^r} \le \mathrm{Err}_N(S^r_{pq}B(\mathbb{Q}^d)) \le c_2 \frac{(\log)^{\frac{q(d-1)}{q-1}}}{N^r},$$

Proof. This follows from (2) and Theorem 1.1 in combination with [T10a, Theorem 6.11].

□

2 The b-adic Haar bases

For some integer $b \ge 2$ a b-adic interval of length b^{-j}, $j \in \mathbb{N}_0$ in $\mathbb{Q}$ is an interval of the form

$$I_{jm} = I^b_{jm} = [b^{-j}m, b^{-j}(m+1))$$

for $m = 0, 1, \ldots, b^j - 1$. For $j \in \mathbb{N}_0$ we divide I_{jm} into b intervals of length b^{-j-1}, i.e. $I^k_{jm} = I^{b,k}_{jm} = I_{j+1,bm+k}$, $k = 0, \ldots, b-1$. As an additional notation we put $I^{-1}_{-1,0} = I_{-1,0} = [0,1)$. Let $\mathbb{D}_j = \{0, 1, \ldots, b^j - 1\}$ and $\mathbb{B}_j = \{1, \ldots, b-1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{0\}$ and $\mathbb{B}_{-1} = \{1\}$. The b-adic Haar functions $h_{jml} = h^b_{jml}$, have support in I_{jm}. For any $j \in \mathbb{N}_0$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$ and any $k = 0, \ldots, b-1$ the value of h_{jml} in I^k_{jm} is $e^{\frac{2\pi i}{b} lk}$. We denote the indicator function of $I_{-1,0}$ by $h_{-1,0,1}$. Let $\mathbb{N}_{-1} = \{-1, 0, 1, 2, \ldots\}$. The functions h_{jml}, $j \in \mathbb{N}_{-1}$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$ are called b-adic Haar system. Normalized in $L_2(\mathbb{Q})$ we obtain the orthonormal b-adic Haar basis of $L_2(\mathbb{Q})$. The proof of this fact can be found in [RW98].

For $j = (j_1, \ldots, j_d) \in \mathbb{N}^d_{-1}$, $m = (m_1, \ldots, m_d) \in \mathbb{D}_j := \mathbb{D}_{j_1} \times \ldots \times \mathbb{D}_{j_d}$ and $l = (l_1, \ldots, l_d) \in \mathbb{B}_j := \mathbb{B}_{j_1} \times \ldots \times \mathbb{B}_{j_d}$, the Haar function h_{jml} is given as the tensor product $h_{jml}(x) = h_{j_1,m_1,l_1}(x_1) \ldots h_{j_d,m_d,l_d}(x_d)$ for $x = (x_1, \ldots, x_d) \in \mathbb{Q}^d$. We will call $I_{jm} = I_{j_1,m_1} \times \ldots \times I_{j_d,m_d}$ b-adic boxes. For $k = (k_1, \ldots, k_d)$ where $k_i \in \{0, \ldots, b-1\}$ for $j_i \in \mathbb{N}_0$ and $k_i = -1$ for $j_i = -1$ we put $I^k_{jm} = I^{k_1}_{j_1 m_1} \times \ldots \times I^{k_d}_{j_d m_d}$. The functions h_{jml}, $j \in \mathbb{N}^d_{-1}$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$ are called d-dimensional b-adic Haar system. Normalized in $L_2(\mathbb{Q}^d)$ we obtain the orthonormal b-adic Haar basis of $L_2(\mathbb{Q}^d)$.

For any function $f \in L_2(\mathbb{Q}^d)$ we have by Parseval's equation

$$\|f|L_2\|^2 = \sum_{j \in \mathbb{N}^d_{-1}} b^{\max(0,j_1)+\ldots+\max(0,j_d)} \sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\mu_{jml}|^2. \tag{3}$$

where

$$\mu_{jml} = \mu_{jml}(f) = \int_{\mathbb{Q}^d} f(x) h_{jml}(x)\,\mathrm{d}x \tag{4}$$

are the b-adic Haar coefficients of f.

Our goal is to combine the b-adic Haar basis method with Triebel's theory in Besov spaces. We generalize [T10a, Theorem 2.41] for b-adic Haar systems in the d-dimensional unit cube. So we characterize Besov spaces $S^r_{pq}B(\mathbb{Q}^d)$ with dominating mixed smoothness.

3 Characterization for Besov spaces with dominating mixed smoothness

Theorem 3.1. *Let $0 < p, q \leq \infty$ and $\frac{1}{p} - 1 < r < \min(\frac{1}{p}, 1)$. Let $f \in D'(\mathbb{Q}^d)$. Then $f \in S^r_{pq}B(\mathbb{Q}^d)$ if and only if it can be represented as*

$$f = \sum_{j \in \mathbb{N}^d_{-1}} \sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} \mu_{jml}\, b^{\max(0,j_1)+\ldots+\max(0,j_d)} h_{jml} \tag{5}$$

for some sequence (μ_{jml}) satisfying

$$\left(\sum_{j \in \mathbb{N}^d_{-1}} b^{(j_1+\ldots+j_d)(r-\frac{1}{p}+1)q} \left(\sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\mu_{jml}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty, \tag{6}$$

where the convergence is unconditional in $D'(\mathbb{Q}^d)$ and in any $S^\rho_{pq}B(\mathbb{Q}^d)$ with $\rho < r$. This representation of f is unique with the b-adic Haar coefficients

$$\mu_{jml} = \mu^b_{jml}(f) = \int_{\mathbb{Q}^d} f(x) h_{jml}(x) dx.$$

The expression (6) additionally delivers an equivalent quasi-norm on $S^r_{pq}B(\mathbb{Q}^d)$.

The definition of the spaces $S^r_{pq}B(\mathbb{Q}^d)$ was dyadic therefore, making it difficult to gain any b-adic results. Hence, we have to change the base first.

Let $\varphi_0 \in \mathcal{S}(\mathbb{R})$ satisfy $\varphi_0(t) = 1$ for $|t| \leq 1$ and $\varphi_0(t) = 0$ for $|t| > \frac{b+1}{b}$. Let

$$\varphi_k(t) = \varphi_0(b^{-k}t) - \varphi_0(b^{-k+1}t)$$

where $t \in \mathbb{R}$, $k \in \mathbb{N}$ and

$$\varphi_k(t) = \varphi_{k_1}(t_1) \ldots \varphi_{k_d}(t_d)$$

where $k = (k_1, \ldots, k_d) \in \mathbb{N}^d_0$, $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$. The functions φ_k are a b-adic

resolution of unity since

$$\sum_{k\in\mathbb{N}_0^d}\varphi_k(x)=1$$

for all $x\in\mathbb{R}^d$. The functions $\mathcal{F}^{-1}(\varphi_k\mathcal{F}f)$ are entire analytic functions for any $f\in\mathcal{S}'(\mathbb{R}^d)$. Let $0<p,q\le\infty$ and $r\in\mathbb{R}$. The b-adic Besov space with dominating mixed smoothness $S^r_{pq}B^b(\mathbb{R}^d)$ consists of all $f\in\mathcal{S}'(\mathbb{R}^d)$ with finite quasi-norm

$$\left\|f|S^r_{pq}B^b(\mathbb{R}^d)\right\|=\left(\sum_{k\in\mathbb{N}_0^d}b^{r(k_1+\ldots+k_d)q}\left\|\mathcal{F}^{-1}(\varphi_k\mathcal{F}f)|L_p(\mathbb{R}^d)\right\|^q\right)^{\frac{1}{q}}$$

with the usual modification if $q=\infty$. We will first prove that the b-adic norm is equivalent to the dyadic norm. Then we will be able to apply Triebel's ideas for the proof of the theorem. To prove the equivalence, we prove the equivalence of the b-adic and the $(b+1)$-adic norms. Let the functions φ_k be a b-adic resolution of unity and the functions ψ_k a $(b+1)$-adic resolution of unity. We observe that

$$\operatorname{supp}\varphi_k\subset[b^{k-1},b^{k+1}]\text{ and }\operatorname{supp}\psi_k\subset[(b+1)^{k-1},(b+1)^{k+1}].$$

Now we check that for every $j\in\mathbb{N}_0$ there are at most 2 such $k\in\mathbb{N}_0$ that $[b^{k-1},b^{k+1}]\subset[(b+1)^{j-1},(b+1)^{j+1}]$. But this is easy since $(b+1)^{j-1}\le b^{k-1}$ and $b^{k+1}\le(b+1)^{j+1}$ is equivalent to

$$(j-1)\frac{\log(b+1)}{\log(b)}+1\le k\le(j+1)\frac{\log(b+1)}{\log(b)}-1. \tag{7}$$

The fact that the cardinality of the set of such k is at most 2 follows from

$$2\frac{\log(b+1)}{\log(b)}-2<2$$

which is equivalent to

$$\frac{\log(b+1)}{\log(b)}<2$$

which is equivalent to $0<b^2-b-1$ which is clearly satisfied since $b\ge 2$. Therefore, we know that for every j there are not more than two k such that, $\operatorname{supp}\varphi_k\subset\operatorname{supp}\psi_j$. For every $j\in\mathbb{N}_0$ we denote by $\Lambda(j)$ the set of such k that $\operatorname{supp}\varphi_k\cap\operatorname{supp}\psi_j\ne\emptyset$. The cardinality of such sets is at most 6 and for sure they are not empty. Conversely, for every $k\in N_0$ there are at most 3 such $j\in\mathbb{N}_{-1}$ that

$\operatorname{supp}\varphi_k \cap \operatorname{supp}\psi_j \neq \emptyset$. We denote by $\Omega(k)$ the set of such j. Additionally, we put for $j \in N_0^d$

$$\Lambda(j) = \Lambda(j_1) \times \ldots \times \Lambda(j_d)$$

and for $k \in \mathbb{N}_{-1}^d$

$$\Omega(k) = \Omega(k_1) \times \ldots \times \Omega(k_d).$$

Hence, for all $x \in \mathbb{R}^d$ we have

$$\varphi_k(x) = \varphi_k(x) \sum_{j \in \Omega(k)} \psi_j(x)$$

and

$$\psi_j(x) = \psi_j(x) \sum_{k \in \Lambda(j)} \varphi_k(x).$$

Now let $j, k \in \mathbb{N}_0^d$ then we have

$$\mathcal{F}^{-1}(\varphi_k \mathcal{F} f) = \sum_{j \in \Omega(k)} \mathcal{F}^{-1}\left(\varphi_k \mathcal{F}\left(\mathcal{F}^{-1}(\psi_j \mathcal{F} f)\right)\right)$$

and

$$\mathcal{F}^{-1}(\psi_j \mathcal{F} f) = \sum_{k \in \Lambda(j)} \mathcal{F}^{-1}\left(\psi_j \mathcal{F}\left(\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)\right)\right).$$

From the d-dimensional version of [ST87, Theorem 1.8.3] (generalization straightforward) with $b_1 = b^{k_1+2}, \ldots, b_d = b^{k_d+2}$ we get (for a constant $c > 0$)

$$\begin{aligned}
&\left\| \mathcal{F}^{-1}\left(\varphi_k \mathcal{F}\left(\mathcal{F}^{-1}(\psi_j \mathcal{F} f)\right)\right) |L_p(\mathbb{R}^d)\right\| \\
&\qquad \leq c \left\| \varphi_k(b^{k+(2,\ldots,2)} \cdot) | S_2^l W(\mathbb{R}^d)\right\| \left\| \mathcal{F}^{-1}(\psi_j \mathcal{F} f) | L_p(\mathbb{R}^d)\right\| \\
&\qquad \leq c_1 \prod_{i=1}^d \left\| \varphi_{k_i}(b^{k_i+2} \cdot) | W_2^l(\mathbb{R})\right\| \left\| \mathcal{F}^{-1}(\psi_j \mathcal{F} f) | L_p(\mathbb{R}^d)\right\|.
\end{aligned}$$

Since $\varphi_{k_i} \in \mathcal{S}(\mathbb{R})$ there exists a constant $c_2 > 0$ such that, for all i we have

$$\left\| \varphi_{k_i}(b^{k_i+2} \cdot) | W_2^l(\mathbb{R})\right\| \leq c_2.$$

Consequently, we get

$$\left\| \mathcal{F}^{-1}\left(\varphi_k \mathcal{F}\left(\mathcal{F}^{-1}(\psi_j \mathcal{F} f)\right)\right) |L_p(\mathbb{R}^d)\right\| \leq c_3 \left\| \mathcal{F}^{-1}(\psi_j \mathcal{F} f) | L_p(\mathbb{R}^d)\right\|$$

for $j \in \Omega(k)$ and analogously (using [ST87, Theorem 1.8.3] with $b_1 = (b+1)^{j_1+2}, \ldots, b_d = (b+1)^{j_d+2}$)

$$\left\| \mathcal{F}^{-1}\left(\psi_j \mathcal{F}\left(\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)\right)\right) | L_p(\mathbb{R}^d) \right\| \leq c_4 \left\| \mathcal{F}^{-1}(\varphi_k \mathcal{F} f) | L_p(\mathbb{R}^d) \right\|$$

for $k \in \Lambda(j)$. So we have proved for every $k \in \mathbb{N}_0^d$ that

$$\left\| \mathcal{F}^{-1}\left(\varphi_k \mathcal{F} f\right) | L_p(\mathbb{R}^d) \right\| \leq c \sum_{j \in \Omega(k)} \left\| \mathcal{F}^{-1}(\psi_j \mathcal{F} f) | L_p(\mathbb{R}^d) \right\|.$$

Multiplying with $b^{r(k_1+\ldots+k_d)q}$ and summing over k will give us on the left side $\left\| \cdot | S^r_{pq} B^b(\mathbb{R}^d) \right\|$. On the right side we get at most 3 identical summands which we can incorporate into the constant. The norming factor can be easily estimated with a constant since the difference of j and k is limited by (7). Conversely, we have for every $j \in \mathbb{N}_0^d$

$$\left\| \mathcal{F}^{-1}\left(\psi_j \mathcal{F} f\right) | L_p(\mathbb{R}^d) \right\| \leq c \sum_{k \in \Lambda(k)} \left\| \mathcal{F}^{-1}(\varphi_k \mathcal{F} f) | L_p(\mathbb{R}^d) \right\|.$$

Multiplying with $(b+1)^{r(j_1+\ldots+j_d)q}$ and summing over j will give us on the left side $\left\| \cdot | S^r_{pq} B^{b+1}(\mathbb{R}^d) \right\|$. On the right side we get at most 6 identical summands which we can incorporate into the constant. The same applies again to the norming factor.

Now we can prove the theorem following closely the original proof from [T10a]. First, one assumes for $\max(\frac{1}{p}, 1) - 1 < r < \min(\frac{1}{p}, 1)$ that the function f is given in the form

$$f = \sum_{j \in \mathbb{N}_0^d} b^{j_1+\ldots+j_d} \sum_{m \in \mathbb{D}_j} \mu_{jm} \, \chi_{jm} \tag{8}$$

where χ_{jm}, $j \in \mathbb{N}_0, m \in \mathbb{D}_j$ are the characteristic functions of the b-adic boxes I_{jm} and the sequence μ_{jm} satisfies

$$\left(\sum_{j \in \mathbb{N}_0^d} b^{(j_1+\ldots+j_d)(r-\frac{1}{p}+1)q} \left(\sum_{m \in \mathbb{D}_j} |\mu_{jm}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty.$$

Then analogously to [T10a, Proposition 2.34] one proves that f belongs to $S^r_{pq} B^b(\mathbb{Q}^d)$ and therefore to $S^r_{pq} B(\mathbb{Q}^d)$. To prove this let ψ_M, ψ_F be real compactly supported L_2-normed b-adic Daubechies wavelets on $\mathbb{R}$ analogous to [T10a, (1.55–1.56)] and

according to [RW98, Theorem 5.1]. We then expand $\chi_{j_1m_1}(x_1), \dots, \chi_{j_dm_d}(x_d)$ into the wavelet representation according to [T10a, (2.51–2.53)] and insert $\chi_{jm}(x) = \chi_{j_1m_1}(x_1) \cdot \ldots \cdot \chi_{j_dm_d}(x_d)$ into (8). We split the resulting expansions as in [T10a, (2.56–2.60)]. Then we have 2^d terms sorted into the cases $(j_1 \geq k_1, \dots, j_d \geq k_d), \dots, (j_1 < k_1, \dots, j_d < k_d)$. The index $k = (k_1, \dots, k_d)$ is according to [T10a, (2.51)]. We get a b-adic version of [T10a, (2.54)] and [T10a, (2.55)]. This guarantees counterparts of [T10a, (2.62–2.66)] and [T10a, (2.73–2.74)]. This observation leads to the norm estimate of the lemma and therefore prooves it. The next step is to estimate

$$\left\| f | S^r_{pq} B(\mathbb{Q}^d) \right\| \geq c \left(\sum_{j \in \mathbb{N}^d_{-1}} b^{(j_1 + \ldots + j_d)(r - \frac{1}{p} + 1)q} \left(\sum_{m \in \mathbb{D}_j, \, l \in \mathbb{B}_j} |\mu_{jml}(f)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{9}$$

for all $f \in S^r_{pq}B(\mathbb{Q}^d)$ analogously to [T10a, Proposition 2.37] (b-adic) where $\mu_{jml}(f)$ is the sequence of the b-adic Haar coefficients. Finally, one gets a counterpart to [T10a, Proposition 2.38] therefore proving the theorem of this section. To do so, we respresent

$$\begin{aligned} h_{jml} &= \sum_{k=0}^{b-1} e^{\frac{2\pi i}{b} kl} \chi_{j+1, bm+k}, \\ h_{-1,0,1} &= \chi_{0,0}. \end{aligned}$$

Then every function represented as in (5) can be represented as in (8) and therefore belongs to $S^r_{pq}B(\mathbb{Q}^d)$. Conversely, every $f \in S^r_{pq}B(\mathbb{Q}^d)$ gives the estimation (9) while the representability (5) follows from the fact that the b-adic Haar system is an orthonormal basis in $L_2(\mathbb{Q}^d)$. Therefore, one obtains the equivalence of the norms. All further technicalities can be found in the proof of [T10a, Theorem 2.9] and the references given there. The unconditionality is clear in view of (6) The assertion can be obtained for $1 < p, q \leq \infty$ with $\frac{1}{p} - 1 < r < 0$ as explained in Step 2 of the proof of [T10a, Proposition 2.38]. It is also explained there how to prove the generalization of the duality. [T10a, Theorem 1.20] is here helpful as well. The remaining cases with $q < \infty$ can be obtained by real interpolation as explained in Step 3 of the proof of [T10a, Proposition 2.38] (with higher dimension not changing anything). All other cases $1 < p < \infty, \frac{1}{p} - 1 < r \leq 0, q = \infty$ can be solved by duality as well.

4 The Haar coefficients of the generalized Hammersley type point sets

Before we can compute the Haar coefficients we need some short calculations. We omit the proofs since they are nothing further but easy exercises.

Lemma 4.1. *For any integer $b \geq 2$ and for any $l \in \{1, \dots, b-1\}$ we have*

$$\sum_{k=1}^{b-1} k e^{\frac{2\pi i}{b} lk} = \frac{b}{e^{\frac{2\pi i}{b} l} - 1} = \sum_{k=0}^{b-2} \sum_{r=k+1}^{b-1} e^{\frac{2\pi i}{b} rl}.$$

Lemma 4.2. *Let $f(x) = x_1 x_2$ for $x = (x_1, x_2) \in \mathbb{Q}^2$. Let $j \in \mathbb{N}_{-1}^2$, $m \in \mathbb{D}_j, l \in \mathbb{B}_j$ and let μ_{jml} be the b-Haar coefficient of f. Then*

(i) If $j = (j_1, j_2) \in \mathbb{N}_0^2$ then

$$\mu_{jml} = \frac{b^{-2j_1 - 2j_2 - 2}}{(e^{\frac{2\pi i}{b} l_1} - 1)(e^{\frac{2\pi i}{b} l_2} - 1)}.$$

(ii) If $j = (j_1, -1)$ with $j_1 \in \mathbb{N}_0$ then

$$\mu_{jml} = \frac{1}{2} \frac{b^{-2j_1 - 1}}{e^{\frac{2\pi i}{b} l_1} - 1}.$$

(iii) If $j = (-1, j_2)$ with $j_2 \in \mathbb{N}_0$ then

$$\mu_{jml} = \frac{1}{2} \frac{b^{-2j_2 - 1}}{e^{\frac{2\pi i}{b} l_2} - 1}.$$

(iv) If $j = (-1, -1)$ then $\mu_{jml} = \frac{1}{4}$.

Lemma 4.3. *Let $z = (z_1, z_2) \in \mathbb{Q}^2$ and $f(x) = \mathbb{1}_{C_z}(x)$ for $x = (x_1, x_2) \in \mathbb{Q}^2$. Let $j \in \mathbb{N}_{-1}^2$, $m \in \mathbb{D}_j, l \in \mathbb{B}_j$ and let μ_{jml} be the Haar coefficient of f. Then $\mu_{jml} = 0$ whenever z is not contained in the interior of the b-adic box I_{jm} supporting the functions h_{jml}. If z is contained in the interior of I_{jm} then*

(i) If $j = (j_1, j_2) \in \mathbb{N}_0^2$ then there is a $k = (k_1, k_2)$ with $k_1, k_2 \in \{0, 1, \dots, b-1\}$

such that z is contained in I_{jm}^{k}. Then

$$\mu_{jml} = b^{-j_1-j_2-2}\left[(bm_1+k_1+1-b^{j_1+1}z_1)e^{\frac{2\pi i}{b}k_1l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b}r_1l_1}\right] \times \\ \times \left[(bm_2+k_2+1-b^{j_2+1}z_2)e^{\frac{2\pi i}{b}k_2l_2} + \sum_{r_2=k_2+1}^{b-1} e^{\frac{2\pi i}{b}r_2l_2}\right].$$

(ii) If $j=(j_1,-1)$ with $j_1\in\mathbb{N}_0$ then there is a $k_1\in\{0,1,\dots,b-1\}$ such that z is contained in $I_{jm}^{k_1}$. Then

$$\mu_{jml} = b^{-j_1-1}\left[(bm_1+k_1+1-b^{j_1+1}z_1)e^{\frac{2\pi i}{b}k_1l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b}r_1l_1}\right](1-z_2).$$

(iii) If $j=(-1,j_2)$ with $j_2\in\mathbb{N}_0$ then there is a $k_2\in\{0,1,\dots,b-1\}$ such that z is contained in $I_{jm}^{k_2}$. Then

$$\mu_{jml} = b^{-j_2-1}(1-z_1)\left[(bm_2+k_2+1-b^{j_2+1}z_2)e^{\frac{2\pi i}{b}k_2l_2} + \sum_{r_2=k_2+1}^{b-1} e^{\frac{2\pi i}{b}r_2l_2}\right].$$

(iv) If $j=(-1,-1)$ then $\mu_{jml}=(1-z_1)(1-z_2)$.

The following lemmas are the last step in the computation of the Haar coefficients.

Lemma 4.4. *Let $j\in\mathbb{N}_0^2$, $m\in\mathbb{D}_j, l\in\mathbb{B}_j$ such that $j_1+j_2<n-1$. Then*

$$\sum_{z\in\mathcal{R}_n\cap I_{jm}}\left[(bm_1+k_1+1-b^{j_1+1}z_1)e^{\frac{2\pi i}{b}k_1l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b}r_1l_1}\right] \times \\ \times \left[(bm_2+k_2+1-b^{j_2+1}z_2)e^{\frac{2\pi i}{b}k_2l_2} + \sum_{r_2=k_2+1}^{b-1} e^{\frac{2\pi i}{b}r_2l_2}\right] \\ = \frac{b^{n-j_1-j_2}\pm b^{j_1+j_2-n+2}}{(e^{\frac{2\pi i}{b}l_1}-1)(e^{\frac{2\pi i}{b}l_2}-1)}.$$

By the sign $\pm$ in the numerator we mean either $+$ or $-$ depending on j.

Proof. Let $z\in I_{jm}$. Then there is a $k\in\{0,1,\dots,b-1\}^2$ such that $z\in I_{jm}^{k}$. We

have $0 \le m_i < b^{j_i}$, $i = 1, 2$. Hence we can expand m_i in base b as

$$m_i = b^{j_i-1} m_1^{(i)} + b^{j_i-2} m_2^{(i)} + \ldots + m_{j_i}^{(i)}.$$

Since $z \in \mathcal{R}_n \cap I_{jm}^k$ we have

$$b^{-j_1-1}(bm_1 + k_1) \le \frac{t_n}{b} + \frac{t_{n-1}}{b^2} + \ldots + \frac{t_1}{b^n} < b^{-j_1-1}(bm_1 + k_1 + 1).$$

Inserting the expansion of m_1 in the last inequality gives us

$$\begin{aligned} \frac{m_1^{(1)}}{b} + \frac{m_2^{(1)}}{b^2} + \ldots + \frac{m_{j_1}^{(1)}}{b^{j_1}} + \frac{k_1}{b^{j_1+1}} &\le \frac{t_n}{b} + \frac{t_{n-1}}{b^2} + \ldots + \frac{t_1}{b^n} \\ &< \frac{m_1^{(1)}}{b} + \frac{m_2^{(1)}}{b^2} + \ldots + \frac{m_{j_1}^{(1)}}{b^{j_1}} + \frac{k_1 + 1}{b^{j_1+1}}. \end{aligned}$$

Analogously we have

$$b^{-j_2-1}(bm_2 + k_2) \le \frac{s_1}{b} + \frac{s_2}{b^2} + \ldots + \frac{s_n}{b^n} < b^{-j_2-1}(bm_2 + k_2 + 1).$$

Hence

$$\begin{aligned} \frac{m_1^{(2)}}{b} + \frac{m_2^{(2)}}{b^2} + \ldots + \frac{m_{j_2}^{(2)}}{b^{j_2}} + \frac{k_2}{b^{j_2+1}} &\le \frac{s_1}{b} + \frac{s_2}{b^2} + \ldots + \frac{s_n}{b^n} \\ &< \frac{m_1^{(2)}}{b} + \frac{m_2^{(2)}}{b^2} + \ldots + \frac{m_{j_2}^{(2)}}{b^{j_2}} + \frac{k_2 + 1}{b^{j_2+1}}. \end{aligned}$$

So one gets a characterization of the fact that $z \in \mathcal{R}_n \cap I_{jm}^k$ in the form

$$t_n = m_1^{(1)},\ t_{n-1} = m_2^{(1)},\ \ldots,\ t_{n-j_1+1} = m_{j_1}^{(1)},\ t_{n-j_1} = k_1$$

and

$$s_1 = m_1^{(2)},\ s_2 = m_2^{(2)},\ \ldots,\ s_{j_2} = m_{j_2}^{(2)},\ s_{j_2+1} = k_2.$$

Hence $t_1, t_2, \ldots, t_{j_2}$ and $t_{n-j_1+1}, \ldots, t_{n-1}, t_n$ are determined by the condition $z \in \mathcal{R}_n \cap I_{jm}$ and t_{n-j_1} and t_{j_2+1} are determined by $k = (k_1, k_2)$ for which $z \in I_{jm}^k$ while $t_{j_2+2}, \ldots, t_{n-j_1-1} \in \{0, 1, \ldots, b-1\}$ can be chosen arbitrarily. Then we calculate

$$\begin{aligned} &bm_1 + k_1 + 1 - b^{j_1+1} z_1 \\ &= 1 + b^{j_1} t_n + b^{j_1-1} t_{n-1} + \ldots + b t_{n-j_1+1} + t_{n-j_1} \end{aligned}$$

$$
\begin{aligned}
&\quad - b^{j_1} t_n - b^{j_1-1} t_{n-1} - \ldots - b^{j_1-n+1} t_1 \\
&= 1 - b^{-1} t_{n-j_1-1} - \ldots - b^{j_1-n+1} t_1 \\
&= 1 - b^{-1} t_{n-j_1-1} - \ldots - b^{j_1+j_2-n+2} t_{j_2+2} - b^{j_1+j_2-n+1} t_{j_2+1} - \varepsilon_1
\end{aligned}
$$

where

$$\varepsilon_1 = b^{j_1+j_2-n} t_{j_2} + \ldots + b^{j_1-n+1} t_1$$

and

$$
\begin{aligned}
&bm_2 + k_2 + 1 - b^{j_2+1} z_2 \\
&= 1 + b^{j_2} s_1 + b^{j_2-1} s_2 + \ldots + b s_{j_2} + s_{j_2+1} \\
&\quad - b^{j_2} s_1 - b^{j_2-1} s_2 - \ldots - b^{j_2-n+1} s_n \\
&= 1 - b^{-1} s_{j_2+2} - \ldots - b^{j_2-n+1} s_n \\
&= 1 - b^{-1} s_{j_2-2} - \ldots - b^{j_1+j_2-n+2} s_{n-j_1-1} - b^{j_1+j_2-n+1} s_{n-j_1} - \varepsilon_2
\end{aligned}
$$

where

$$\varepsilon_2 = b^{j_1+j_2-n} s_{n-j_1+1} + \ldots + b^{j_1-n+1} s_n.$$

This means that

$$bm_1 + k_1 + 1 - b^{j_1+1} z_1 = h b^{j_1+j_2-n+2} - b^{j_1+j_2-n+1} t_{j_2+1} - \varepsilon_1$$

for $h = 1, 2, \ldots, b^{n-j_1-j_2-2}$. It is clear that there must be some permutation σ of $\{1, 2, \ldots, b^{n-j_1-j_2-2}\}$ such that

$$bm_2 + k_2 + 1 - b^{j_2+1} z_2 = \sigma(h) b^{j_1+j_2-n+2} - b^{j_1+j_2-n+1} s_{n-j_1} - \varepsilon_2.$$

We abbreviate $X = n - j_1 - j_2 - 2$. Then

$$
\begin{aligned}
&\sum_{z \in \mathcal{R}_n \cap I_{jm}} \left[(bm_1 + k_1 + 1 - b^{j_1+1} z_1) e^{\frac{2\pi i}{b} k_1 l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} \right] \times \\
&\qquad\qquad \times \left[(bm_2 + k_2 + 1 - b^{j_2+1} z_2) e^{\frac{2\pi i}{b} k_2 l_2} + \sum_{r_2=k_2+1}^{b-1} e^{\frac{2\pi i}{b} r_2 l_2} \right] \\
&= \sum_{k_1=0}^{b-1} \sum_{k_2=0}^{b-1} \sum_{z \in \mathcal{R}_n \cap I_{jm}^k} [\ldots] \times [\ldots]
\end{aligned}
$$

$$= \sum_{k_1=0}^{b-1} \sum_{k_2=0}^{b-1} \sum_{h=1}^{b^X} \left[\left(hb^{-X} - b^{-X-1} t_{j_2+1} - \varepsilon_1 \right) e^{\frac{2\pi i}{b} k_1 l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} \right] \times$$
$$\times \left[\left(\sigma(h) b^{-X} - b^{-X-1} s_{n-j_1} - \varepsilon_2 \right) e^{\frac{2\pi i}{b} k_2 l_2} + \sum_{r_2=k_2+1}^{b-1} e^{\frac{2\pi i}{b} r_2 l_2} \right].$$

After having expanded the product and changed the order of summation we analyze the summands separately in a fitting order. We recall that s_{n-j_1} depends on k_1 and t_{j_2+1} depends on k_2. Except the last two, all summands are equal to zero because each has the sum of unity roots as a factor. The summands are the following

$$\sum_{h=1}^{b^X} \left(hb^{-X} - \varepsilon_1 \right) \left(\sigma(h) b^{-X} - \varepsilon_2 \right) \sum_{k_1=0}^{b-1} e^{\frac{2\pi i}{b} k_1 l_1} \sum_{k_2=0}^{b-1} e^{\frac{2\pi i}{b} k_2 l_2} = 0,$$
$$- \sum_{h=1}^{b^X} \left(hb^{-X} - \varepsilon_1 \right) b^{-X-1} \sum_{k_1=0}^{b-1} s_{n-j_1} e^{\frac{2\pi i}{b} k_1 l_1} \sum_{k_2=0}^{b-1} e^{\frac{2\pi i}{b} k_2 l_2} = 0,$$
$$- \sum_{h=1}^{b^X} \left(\sigma(h) b^{-X} - \varepsilon_2 \right) b^{-X-1} \sum_{k_2=0}^{b-1} t_{j_2+1} e^{\frac{2\pi i}{b} k_2 l_2} \sum_{k_1=0}^{b-1} e^{\frac{2\pi i}{b} k_1 l_1} = 0,$$
$$\sum_{h=1}^{b^X} \left(hb^{-X} - \varepsilon_1 \right) \sum_{k_2=0}^{b-1} \sum_{r_2=k_2+1}^{b-1} e^{\frac{2\pi i}{b} r_2 l_2} \sum_{k_1=0}^{b-1} e^{\frac{2\pi i}{b} k_1 l_1} = 0,$$
$$\sum_{h=1}^{b^X} \left(\sigma(h) b^{-X} - \varepsilon_2 \right) \sum_{k_1=0}^{b-1} \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} \sum_{k_2=0}^{b-1} e^{\frac{2\pi i}{b} k_2 l_2} = 0,$$
$$- \sum_{h=1}^{b^X} b^{-X-1} \sum_{k_1=0}^{b-1} \sum_{r_1=k_1+1}^{b-1} s_{n-j_1} e^{\frac{2\pi i}{b} r_1 l_1} \sum_{k_2=0}^{b-1} e^{\frac{2\pi i}{b} k_2 l_2} = 0,$$
$$- \sum_{h=1}^{b^X} b^{-X-1} \sum_{k_2=0}^{b-1} \sum_{r_2=k_2+1}^{b-1} t_{j_2+1} e^{\frac{2\pi i}{b} r_2 l_2} \sum_{k_1=0}^{b-1} e^{\frac{2\pi i}{b} k_1 l_1} = 0,$$
$$\sum_{h=1}^{b^X} \sum_{k_1=0}^{b-1} \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} \sum_{k_2=0}^{b-1} \sum_{r_2=k_2+1}^{b-1} e^{\frac{2\pi i}{b} r_2 l_2} = \frac{b^{n-j_1-j_2}}{(e^{\frac{2\pi i}{b} l_1} - 1)(e^{\frac{2\pi i}{b} l_2} - 1)}$$

by Lemma 4.1. Finally, the last summand is

$$\sum_{h=1}^{b^X} \sum_{k_1=0}^{b-1} \sum_{k_2=0}^{b-1} b^{-X-1} t_{j_2+1} b^{-X-1} s_{n-j_1} e^{\frac{2\pi i}{b} k_1 l_1} e^{\frac{2\pi i}{b} k_2 l_2}$$

$$= b^{j_1+j_2-n} \sum_{k_1=0}^{b-1} s_{n-j_1} e^{\frac{2\pi i}{b} k_1 l_1} \sum_{k_2=0}^{b-1} t_{j_2+1} e^{\frac{2\pi i}{b} k_2 l_2}.$$

We know that $t_{n-j_1} = k_1$ and that either $s_i = t_i$ or $s_i = b-1-t_i$ for all $i = 1, \ldots, n$. Hence s_{n-j_1} is either k_1 or $b-1-k_1$. Since

$$\sum_{k_1=0}^{b-1} (b-1) e^{\frac{2\pi i}{b} k_1 l_1} = 0$$

we have

$$\sum_{k_1=0}^{b-1} s_{n-j_1} e^{\frac{2\pi i}{b} k_1 l_1} = \pm \frac{b}{e^{\frac{2\pi i}{b} l_1} - 1} \tag{10}$$

using Lemma 4.1 and the sign depends on j_1. Also we know that $s_{j_2+1} = k_2$ and that either $s_{j_2+1} = t_{j_2+1}$ or $s_{j_2} = b-1-t_{j_2+1}$. Hence

$$\sum_{k_2=0}^{b-1} t_{j_2+1} e^{\frac{2\pi i}{b} k_2 l_2} = \pm \frac{b}{e^{\frac{2\pi i}{b} l_2} - 1}$$

and the sign depends on j_2. So alltogether our last summand is

$$b^{j_1+j_2-n} \frac{\pm b^2}{(e^{\frac{2\pi i}{b} l_1} - 1)(e^{\frac{2\pi i}{b} l_2} - 1)} = \frac{\pm b^{j_1+j_2-n+2}}{(e^{\frac{2\pi i}{b} l_1} - 1)(e^{\frac{2\pi i}{b} l_2} - 1)}$$

and the sign depends on j. Adding both summands which are nonzero gives us the stated result.

□

Lemma 4.5. *Let*

$$x_n := \sum_{t_1,\ldots,t_n=0}^{b-1} \sum_{j=1}^{n} b^{-j} t_j$$

and

$$y_n := \sum_{t_1,\ldots,t_n=0}^{b-1} \sum_{i=1}^{n} b^{i} t_i$$

for any positive integer n. Then

$$x_n = \frac{1}{2}(b^n - 1)$$

and

$$y_n = b^{n+1} x_n = \frac{1}{2} b^{n+1}(b^n - 1).$$

Proof. Clearly, $x_1 = \frac{1}{2}(b-1)$ and inductively

$$\begin{aligned} x_n &= \sum_{t_n} \sum_{t_1,\ldots,t_{n-1}} \sum_{j=1}^{n-1} b^{-j} t_j + b^{-n} \sum_{t_1,\ldots,t_{n-1}} \sum_{t_n} t_n \\ &= b\, x_{n-1} + b^{-n}\, b^{n-1}\, \frac{b\,(b-1)}{2} \\ &= b\, \frac{1}{2}\,(b^{n-1} - 1) + \frac{1}{2}\,(b-1) \\ &= \frac{1}{2}\,(b^n - 1). \end{aligned}$$

One sees that $y_n = b^{n+1} x_n$ simply by checking that

$$\sum_{i=1}^{n} b^i t_i = b^{n+1} \sum_{i=1}^{n} b^{i-n-1} t_i = b^{n+1} \sum_{i=1}^{n} b^{-i} t_{n+1-i}.$$

Summing over $t_1, \ldots, t_n$ will give us y_n on the left side. On the right side it will give us $b^{n+1} x_n$ although the order of the t_i is reversed with respect to the definition of the numbers x_n.

□

We will use this fact that the order of the t_i is irrelevant in further proofs. But not only the order is irrelevant but even the concrete index of the t_j. For example the value of

$$\sum_{t_{n+1},\ldots,t_{2n}=0}^{b-1} \sum_{j=1}^{n} b^{-j} t_{j+n}$$

is the same as the value of x_n.

Lemma 4.6. *Let*

$$z_n := \sum_{t_1,\ldots,t_n=0}^{b-1} \sum_{i,j=1}^{n} b^{i-j} t_i t_j$$

for any positive integer n. Then

$$z_n = \frac{1}{4} b^{2n+1} + \frac{n}{12} b^{n+2} - \frac{1}{2} b^{n+1} - \frac{n}{12} b^n + \frac{1}{4} b.$$

Proof. Clearly, $z_1 = \frac{1}{6}(b-1)b(2b-1) = \frac{1}{3}b^3 - \frac{1}{2}b^2 + \frac{1}{6}b$. Then inductively we get

$$\begin{aligned}
z_n &= \sum_{t_n} \sum_{t_1,\dots,t_{n-1}} \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} b^{i-j} t_i t_j + b^n \sum_{t_n} t_n \sum_{t_1,\dots,t_{n-1}} \sum_{j=1}^{n-1} b^{-j} t_j + \\
&\quad + b^{-n} \sum_{t_n} t_n \sum_{t_1,\dots,t_{n-1}} \sum_{i=1}^{n-1} b^i t_j + \sum_{t_1,\dots,t_{n-1}} \sum_{t_n} t_n^2 \\
&= b\, z_{n-1} + b^n \frac{1}{2}(b-1)\, b\, x_{n-1} + b^{-n}\frac{1}{2}(b-1)\, b\, y_{n-1} + b^{n-1}\frac{1}{6}(b-1)\, b\,(2b-1) \\
&= b\left(\frac{1}{4}b^{2n-1} + \frac{n-1}{12} b^{n+1} - \frac{1}{2} b^n - \frac{n-1}{12} b^{n-1} + \frac{1}{4} b\right) + \\
&\quad + b^n \frac{1}{2}(b-1)\, b \left(\frac{1}{2}(b^{n-1}-1)\right) + b^{-n}\frac{1}{2}(b-1)\, b\left(\frac{1}{2} b^n (b^{n-1}-1)\right) + \\
&\quad + b^{n-1}\frac{1}{6}(b-1)\, b\,(2b-1) \\
&= \frac{1}{4}b^{2n+1} + \frac{n}{12}b^{n+2} - \frac{1}{2}b^{n+1} - \frac{n}{12}b^n + \frac{1}{4}b.
\end{aligned}$$

□

Lemma 4.7. *Let $z = (z_1, z_2)$. Then*

$$\sum_{z\in\mathcal{R}_n} (1-z_1)(1-z_2) = 1 + b^{-n-1} \sum_{t_1,\dots,t_n}^{b-1} \sum_{i,j=1}^{n} b^{i-j} t_i s_j.$$

Proof. We first calculate for some $z \in \mathcal{R}_n$

$$\begin{aligned}
(1-z_1)(1-z_2) &= (1 - b^{-1}t_n - \ldots - b^{-n}t_1)(1 - b^{-1}s_1 - \ldots - b^{-n}s_n) \\
&= 1 - b^{-1}t_n - \ldots - b^{-n}t_1 - b^{-1}s_1 - \ldots - b^{-n}s_n + \\
&\quad + \sum_{i,j=1}^{n} b^{-n+i-j-1} t_i s_j.
\end{aligned}$$

Now we sum over all $z \in \mathcal{R}_n$ which corresponds to summing over all $t_1, \dots, t_n \in \{0, 1, \dots, b-1\}$ and get

$$\begin{aligned}
&\sum_{z\in\mathcal{R}_n\cap I_{(-1,-1),(0,0)}} (1-z_1)(1-z_2) \\
&= \sum_{t_1,\dots,t_n} \left(1 - b^{-1}t_n - \ldots - b^{-n}t_1 - b^{-1}s_1 - \ldots - b^{-n}s_n + b^{-n-1}\sum_{i,j=1}^{n} b^{i-j} t_i s_j\right)
\end{aligned}$$

$$= b^n - b^{-1} b^{n-1} \sum_{t_n=0}^{b-1} t_n - b^{-1} b^{n-1} \sum_{t_1=0}^{b-1} s_1 - \ldots - b^{-n} b^{n-1} \sum_{t_1=0}^{b-1} t_1 - b^{-n} b^{n-1} \sum_{t_n=0}^{b-1} s_n +$$

$$+ b^{-n-1} \sum_{t_1,\ldots,t_n} \sum_{i,j=1}^{n} b^{i-j} t_i s_j$$

$$= b^n - 2\left(b^{n-2} \frac{1}{2}(b-1)\,b + \ldots + b^{-1} \frac{1}{2}(b-1)\,b\right) + b^{-n-1} \sum_{t_1,\ldots,t_n} \sum_{i,j=1}^{n} b^{i-j} t_i s_j$$

$$= b^n - (b-1)(b^{n-1} + \ldots + 1) + b^{-n-1} \sum_{t_1,\ldots,t_n} \sum_{i,j=1}^{n} b^{i-j} t_i s_j$$

$$= 1 + b^{-n-1} \sum_{t_1,\ldots,t_n} \sum_{i,j=1}^{n} b^{i-j} t_i s_j$$

$\square$

Lemma 4.8. *We consider a generalized Hammersley type point set $\mathcal{R}_n$. Then*

$$\sum_{t_1,\ldots,t_n=0}^{b-1} \sum_{i,j=1}^{n} b^{i-j} t_i s_j = \frac{1}{4} b^{2n+1} - \frac{1}{2} b^{n+1} + \frac{1}{4} b + (2a_n - n) \frac{b^2-1}{12} b^n.$$

Proof. For better readability we write a instead of a_n. We can assume that $s_1 = t_1, \ldots, s_a = t_a, s_{a+1} = b-1-t_{a+1}, \ldots, s_n = b-1-t_n$. Otherwise we would have to rename the t_j. This assumption allows us to split the sum in a compact way. So,

$$\sum_{i,j=1}^{n} b^{i-j} t_i s_j = \sum_{i,j=1}^{a} b^{i-j} t_i t_j + \sum_{i=1}^{a} \sum_{j=a+1}^{n} b^{i-j} t_i (b-1-t_j) +$$

$$+ \sum_{i=a+1}^{n} \sum_{j=1}^{a} b^{i-j} t_i t_j + \sum_{i,j=a+1}^{n} b^{i-j} t_i (b-1-t_j)$$

$$= \sum_{i,j=1}^{a} b^{i-j} t_i t_j + (b-1) \sum_{i=1}^{a} \sum_{j=a+1}^{n} b^{i-j} t_i - \sum_{i=1}^{a} \sum_{j=a+1}^{n} b^{i-j} t_i t_j +$$

$$+ \sum_{i=a+1}^{n} \sum_{j=1}^{a} b^{i-j} t_i t_j + (b-1) \sum_{i=a+1}^{n} \sum_{j=a+1}^{n} b^{i-j} t_i - \sum_{i=a+1}^{n} \sum_{j=a+1}^{n} b^{i-j} t_i t_j.$$

Summing over $t_1, \ldots, t_n$ and analyzing every term separately will give us

$$\sum_{t_1,\ldots,t_n} \sum_{i,j=1}^{a} b^{i-j} t_i t_j = b^{n-a} z_a,$$

as well as using $y_n = b^{n+1}x_n$

$$\sum_{t_1,\dots,t_n}(b-1)\sum_{i=1}^{a}\sum_{j=a+1}^{n} b^{i-j}t_i = (b-1)b^{n-a}y_a\sum_{j=a+1}^{n} b^{-j}$$
$$= b^{n+1}x_a(b^{-a}-b^{-n}),$$

and

$$\sum_{t_1,\dots,t_n}\sum_{i=1}^{a}\sum_{j=a+1}^{n} b^{i-j}t_it_j = \sum_{t_1,\dots,t_a}\sum_{i=1}^{a} b^it_i \sum_{t_{a+1},\dots,t_n}\sum_{j=a+1}^{n} b^{-j}t_j$$
$$= y_a\sum_{t_{a+1},\dots,t_n} b^{-a}\sum_{j=a+1}^{n} b^{a-j}t_j = x_ax_{n-a}b,$$

since we have already seen that the indexes of t_j are irrelevant. We also get with a similar argumentation

$$\sum_{t_1,\dots,t_n}\sum_{i=a+1}^{n}\sum_{j=1}^{a} b^{i-j}t_it_j = \sum_{t_1,\dots,t_a}\sum_{j=1}^{a} b^{-j}t_j \sum_{t_{a+1},\dots,t_n}\sum_{i=a+1}^{n} b^it_i$$
$$= \sum_{t_1,\dots,t_a}\sum_{j=1}^{a} b^{-j}t_j \sum_{t_{a+1},\dots,t_n} b^a \sum_{i=a+1}^{n} b^{i-a}t_i = x_ab^ay_{n-a} = x_ax_{n-a}b^{n+1},$$
$$\sum_{t_1,\dots,t_n}(b-1)\sum_{i=a+1}^{n}\sum_{j=a+1}^{n} b^{i-j}t_i = (b-1)b^a\sum_{t_{a+1},\dots,t_n}\sum_{i=a+1}^{n} b^it_i\sum_{j=a+1}^{n} b^{-j}$$
$$= b^ay_{n-a}b^a(b^{-a}-b^{-n}) = x_{n-a}(b^{n+1}-b^{a+1})$$

and

$$\sum_{t_1,\dots,t_n}\sum_{i=a+1}^{n}\sum_{j=a+1}^{n} b^{i-j}t_it_j = b^a\sum_{t_{a+1},\dots,t_n}\sum_{i=a+1}^{n}\sum_{j=a+1}^{n} b^{(i-a)+(a-j)}t_it_j = b^az_{n-a}.$$

So what we have is

$$\sum_{t_1,\dots,t_n}^{b-1}\sum_{i,j=1}^{n} b^{i-j}t_is_j$$
$$= b^{n-a}z_a - b^az_{n-a} + x_ab(b^{n-a}-1) + x_ax_{n-a}b(b^n-1) + x_{n-a}b^{a+1}(b^{n-a}-1).$$

Inserting the values of z_a, z_{n-a}, x_a, and x_{n-a} and simplifying will give us the stated assertion.

□

Proposition 4.9. *Let μ_{jml} be the b-adic Haar coefficients of the discrepancy function of $\mathcal{R}_n$. Then*

$$\mu_{(-1,-1),(0,0),(1,1)} = \frac{1}{4}b^{-2n} + \frac{1}{2}b^{-n} + (2a_n - n)\frac{b^2-1}{12}b^{-n-1}.$$

Proof. Using the last lemma we have

$$\sum_{t_1,\ldots,t_n=0}^{b-1} \sum_{i,j=1}^{n} b^{i-j} t_i s_j = \frac{1}{4}b^{2n+1} - \frac{1}{2}b^{n+1} + \frac{1}{4}b + (2a_n - n)\frac{b^2-1}{12}b^n.$$

Hence using Lemmas 4.2, 4.3 and 4.7

$$\begin{aligned}\mu_{(-1,-1),(0,0),(1,1)} &= b^{-n} \sum_{z\in\mathcal{R}_n} (1-z_1)(1-z_2) - \frac{1}{4} \\ &= b^{-n}\left(1 + b^{-n-1}\left(\frac{1}{4}b^{2n+1} - \frac{1}{2}b^{n+1} + \frac{1}{4}b + (2a_n - n)\,\frac{b^2-1}{12}\,b^n\right)\right) - \frac{1}{4} \\ &= \frac{1}{4}b^{-2n} + \frac{1}{2}b^{-n} + (2a_n - n)\,\frac{b^2-1}{12}\,b^{-n-1}.\end{aligned}$$

□

Lemma 4.10. *Let $j = (j_1, -1)$ for $j_1 \in \mathbb{N}_0$ with $j_1 \le n-1$, $m = (m_1, 0)$ with $0 \le m_1 < b^{j_1}$ and $l = (l_1, 1)$ with $1 \le l_1 < b$. Then*

$$\sum_{z\in\mathcal{R}_n\cap I_{jm}} \left[(bm_1 + k_1 + 1 - b^{j_1+1}z_1)e^{\frac{2\pi i}{b}k_1 l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b}r_1 l_1}\right](1-z_2)$$
$$= \frac{b^{n-j_1}(1-2\varepsilon) \mp b^{j_1-n+1}}{2(e^{\frac{2\pi i}{b}l_1}-1)} + \frac{w_{j_1}}{(e^{\frac{2\pi i}{b}l_1}-1)^2},$$

where w_{j_1} is either $e^{\frac{2\pi i}{b}l_1}$ or -1, the sign of $\mp$ depends on j_1 and we have $\varepsilon b^{n-j_1} \le b$.

An analogous result holds for $j = (-1, j_2)$ where $j_2 \in \mathbb{N}_0$ with $j_2 \le n-1$, $m = (0, m_2)$ with $0 \le m_2 < b^{j_2}$ and $l = (1, l_2)$ with $1 \le l_2 < b$.

Proof. Let $z \in \mathcal{R}_n \cap I_{jm}$. Then there is a $k = (k_1, -1)$, $k_1 \in \{0, 1, \ldots, b-1\}$ such that, $z \in \mathcal{R}_n \cap I^k_{jm}$. We use the methods from from Lemma 4.4 for the proof. We have

$$bm_1 + k_1 + 1 - b^{j_1+1}z_1 = 1 - b^{-1}t_{n-j_1-1} - \ldots - b^{j_1-n+1}t_1$$

which means that

$$bm_1 + k_1 + 1 - b^{j_1+1} z_1 = h b^{j_1-n+1}$$

for $h = 1, 2, \ldots, b^{n-j_1-1}$. The numbers $t_{n-j_1+1}, \ldots, t_n$ are determined by the condition $z \in \mathcal{R}_n \cap I_{jm}$ and $t_{n-j_1} = k_1$. All other t_j can be chosen arbitrarily. We also have

$$1 - z_2 = 1 - b^{-1} s_1 - \ldots - b^{j_1-n+1} s_{n-j_1-1} - b^{j_1-n} s_{n-j_1} - \varepsilon$$

where $\varepsilon = b^{j_1-n-1} s_{n-j_1+1} + \ldots + b^{-n} s_n$. Clearly, $\varepsilon b^{n-j_1} \leq b$.

So there must be a permutation σ such that

$$1 - z_2 = \sigma(h) b^{j_1-n+1} - b^{j_1-n} s_{n-j_1} - \varepsilon.$$

Hence

$$\sum_{z \in \mathcal{R}_n \cap I_{jm}} \left[(bm_1 + k_1 + 1 - b^{j_1+1} z_1) e^{\frac{2\pi i}{b} k_1 l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} \right] (1 - z_2)$$
$$= \sum_{k_1=0}^{b-1} \sum_{h=1}^{b^{n-j_1-1}} \left[h b^{j_1-n+1} e^{\frac{2\pi i}{b} k_1 l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} \right] (\sigma(h) b^{j_1-n+1} - b^{j_1-n} s_{n-j_1} - \varepsilon)$$

We analyze the summands separately after having expanded the product and changed the order of summation. We have

$$\sum_{h=1}^{b^{n-j_1-1}} h \sigma(h) b^{j_1-n+1} b^{j_1-n+1} \sum_{k_1=0}^{b-1} e^{\frac{2\pi i}{b} k_1 l_1} = 0,$$
$$- \sum_{h=1}^{b^{n-j_1-1}} h b^{j_1-n+1} b^{j_1-n} \sum_{k_1=0}^{b-1} s_{n-j_1} e^{\frac{2\pi i}{b} k_1 l_1}$$
$$= -\frac{1}{2} b^{n-j_1-1} (b^{n-j_1-1} + 1) b^{2j_1-2n+1} \frac{\pm b}{e^{\frac{2\pi i}{b} l_1} - 1} = \mp \frac{b^{j_1-n+1} + 1}{2(e^{\frac{2\pi i}{b} l_1} - 1)},$$

using the equation (10) from the proof for Lemma 4.4,

$$-\varepsilon \sum_{h=1}^{b^{n-j_1-1}} h b^{j_1-n+1} \sum_{k_1=0}^{b-1} e^{\frac{2\pi i}{b} k_1 l_1} = 0,$$

$$\sum_{h=1}^{b^{n-j_1-1}} \sigma(h) b^{j_1-n+1} \sum_{k_1=0}^{b-1} \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1}$$
$$= \frac{1}{2} b^{n-j_1-1}(b^{n-j_1-1}+1) b^{j_1-n+1} \frac{b}{e^{\frac{2\pi i}{b} l_1}-1} = \frac{b^{n-j_1}+b}{2(e^{\frac{2\pi i}{b} l_1}-1)},$$
$$-\varepsilon \sum_{h=1}^{b^{n-j_1-1}} \sum_{k_1=0}^{b-1} \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} = -\varepsilon b^{n-j_1-1} \frac{b}{e^{\frac{2\pi i}{b} l_1}-1} = \frac{-\varepsilon b^{n-j_1}}{e^{\frac{2\pi i}{b} l_1}-1}.$$

and

$$-\sum_{h=1}^{b^{n-j_1-1}} b^{j_1-n} \sum_{k_1=0}^{b-1} s_{n-j_1} \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1}$$

For the last term we use the fact that s_{n-j_1} is either k_1 or $b-1-k_1$. In the first case we have

$$\sum_{k_1=0}^{b-1} k_1 \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1}$$
$$= \sum_{k_1=1}^{b-2} k_1 \frac{1-e^{\frac{2\pi i}{b}(k_1+1) l_1}}{e^{\frac{2\pi i}{b} l_1}-1}$$
$$= \frac{1}{e^{\frac{2\pi i}{b} l_1}-1} \left(\frac{1}{2}(b-2)(b-1) - \sum_{k_1=2}^{b-1} (k_1-1) e^{\frac{2\pi i}{b} k_1 l_1} \right)$$
$$= \frac{1}{e^{\frac{2\pi i}{b} l_1}-1} \left(\frac{1}{2}(b-2)(b-1) - \left(\frac{b}{e^{\frac{2\pi i}{b} l_1}-1} - e^{\frac{2\pi i}{b} l_1} \right) + \left(0-1-e^{\frac{2\pi i}{b} l_1}\right) \right)$$
$$= \frac{1}{e^{\frac{2\pi i}{b} l_1}-1} \left(\frac{b^2-3b}{2} - \frac{b}{e^{\frac{2\pi i}{b} l_1}-1} \right)$$
$$= \frac{(b-3)b}{2(e^{\frac{2\pi i}{b} l_1}-1)} - \frac{b}{(e^{\frac{2\pi i}{b} l_1}-1)^2}.$$

In the other case we have

$$\sum_{k_1=0}^{b-1} (b-1-k_1) \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1}$$
$$= (b-1) \sum_{k_1=0}^{b-1} \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} - \sum_{k_1=0}^{b-1} k_1 \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1}$$

$$= \frac{(b-1)b}{(e^{\frac{2\pi i}{b}l_1}-1)} - \frac{(b-3)b}{2(e^{\frac{2\pi i}{b}l_1}-1)} + \frac{b}{(e^{\frac{2\pi i}{b}l_1}-1)^2}$$
$$= \frac{b(b+1)}{2(e^{\frac{2\pi i}{b}l_1}-1)} + \frac{b}{(e^{\frac{2\pi i}{b}l_1}-1)^2}.$$

So the last term is either

$$\frac{1}{(e^{\frac{2\pi i}{b}l_1}-1)^2} - \frac{b-3}{2(e^{\frac{2\pi i}{b}l_1}-1)}$$

or

$$-\frac{b+1}{2(e^{\frac{2\pi i}{b}l_1}-1)} - \frac{1}{(e^{\frac{2\pi i}{b}l_1}-1)^2}.$$

Now combining the results we get in the case $s_{n-j_1} = k_1$

$$\sum_{z \in \mathcal{R}_n \cap I_{jm}} \left[(bm_1 + k_1 + 1 - b^{j_1+1} z_1) e^{\frac{2\pi i}{b}k_1 l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} \right] (1 - z_2)$$
$$= \frac{b^{n-j_1}(1-2\varepsilon) - b^{j_1-n+1}}{2(e^{\frac{2\pi i}{b}l_1}-1)} + \frac{e^{\frac{2\pi i}{b}l_1}}{(e^{\frac{2\pi i}{b}l_1}-1)^2}$$

while in the case $s_{n-j_1} = b - 1 - k_1$

$$\sum_{z \in \mathcal{R}_n \cap I_{jm}} \left[(bm_1 + k_1 + 1 - b^{j_1+1} z_1) e^{\frac{2\pi i}{b}k_1 l_1} + \sum_{r_1=k_1+1}^{b-1} e^{\frac{2\pi i}{b} r_1 l_1} \right] (1 - z_2)$$
$$= \frac{b^{n-j_1}(1-2\varepsilon) + b^{j_1-n+1}}{2(e^{\frac{2\pi i}{b}l_1}-1)} - \frac{1}{(e^{\frac{2\pi i}{b}l_1}-1)^2}$$

as stated by the lemma.

□

5 Proof of the main result

Proposition 5.1. *Let $\mathcal{R}_n$ be a generalized Hammersley type point set and let μ_{jml} be the b-adic Haar coefficient of the discrepancy function of $\mathcal{R}_n$ for $j \in \mathbb{N}_{-1}^2$, $m \in \mathbb{D}_j$ and $l \in \mathbb{B}_j$. Then*

(i) if $j \in \mathbb{N}_0^2$ and $j_1 + j_2 < n - 1$ then

$$|\mu_{jml}| = \frac{b^{-2n}}{\left|e^{\frac{2\pi i}{b}l_1} - 1\right| \left|e^{\frac{2\pi i}{b}l_2} - 1\right|},$$

(ii) *if* $j \in \mathbb{N}_0^2$*,* $j_1 + j_2 \geq n - 1$ *and* $j_1, j_2 \leq n$ *then* $|\mu_{jml}| \leq cb^{-n-j_1-j_2}$ *for some constant* $c > 0$ *and*

$$|\mu_{jml}| = \frac{b^{-2j_1-2j_2-2}}{\left|e^{\frac{2\pi i}{b}l_1} - 1\right| \left|e^{\frac{2\pi i}{b}l_2} - 1\right|}$$

for all but b^n *coefficients* μ_{jml}*,*

(iii) *if* $j \in \mathbb{N}_0^2$ *and* $j_1 \geq n$ *or* $j_2 \geq n$ *then*

$$|\mu_{jml}| = \frac{b^{-2j_1-2j_2-2}}{\left|e^{\frac{2\pi i}{b}l_1} - 1\right| \left|e^{\frac{2\pi i}{b}l_2} - 1\right|},$$

(iv) *if* $j = (j_1, -1)$ *with* $j_1 \in \mathbb{N}_0$ *and* $j_1 < n$ *then* $|\mu_{jml}| \leq cb^{-n-j_1}$ *for some constant* $c > 0$*,*

(v) *if* $j = (-1, j_2)$ *with* $j_2 \in \mathbb{N}_0$ *and* $j_2 < n$ *then* $|\mu_{jml}| \leq cb^{-n-j_2}$ *for some constant* $c > 0$*,*

(vi) *if* $j = (j_1, -1)$ *with* $j_1 \in \mathbb{N}_0$ *and* $j_1 \geq n$ *then*

$$|\mu_{jml}| = \frac{1}{2} \frac{b^{-2j_1-1}}{\left|e^{\frac{2\pi i}{b}l_1} - 1\right|},$$

(vii) *if* $j = (-1, j_2)$ *with* $j_2 \in \mathbb{N}_0$ *and* $j_2 \geq n$ *then*

$$|\mu_{jml}| = \frac{1}{2} \frac{b^{-2j_2-1}}{\left|e^{\frac{2\pi i}{b}l_2} - 1\right|},$$

(viii) $\left|\mu_{(-1,-1),(0,0),(1,1)}\right| = |\frac{1}{4}b^{-2n} + \left(\frac{1}{2} + (2a_n - n)\frac{b-b^{-1}}{12}\right) b^{-n}|$.

Proof. Let $j \in \mathbb{N}_{-1}^2$ such that $j_1 \geq n$ or $j_2 \geq n$. Then there is no point of $\mathcal{R}_n$ which is contained in the interior of the b-adic box I_{jm}. Thereby (iii), (vi) and (vii) follow from Lemma 4.2 and Lemma 4.3.

The set $\mathcal{R}_n$ contains $N = b^n$ points and, for fixed $j \in \mathbb{N}_{-1}^2$, the interiors of the b-adic boxes I_{jm} are mutually disjoint. Therefore there are no more than b^n b-adic

boxes which contain a point of $\mathcal{R}_n$. This gives us the second part of (ii). The first part of (ii) follows from Lemma 4.2 and Lemma 4.3 because the remaining boxes contain exactly one point of $\mathcal{R}_n$.

The part (i) follows from Lemmas 4.2, 4.3 and 4.4.

The last part is actually Proposition 4.9.

Finally (iv) (and analogously (v)) follows from Lemma 4.10 combined with Lemma 4.2 and Lemma 4.3. We get

$$|\mu_{jml}| = \left| \frac{b^{-n-j_1-1}(w_{j_1} - \varepsilon\, b^{n-j_1}\,(e^{\frac{2\pi i}{b}l_1} - 1))}{(e^{\frac{2\pi i}{b}l_1} - 1)^2} \pm \frac{b^{-2n}}{2(e^{\frac{2\pi i}{b}l_1} - 1)} \right|$$

where w_{j_1} is either $e^{\frac{2\pi i}{b}l_1}$ or -1. Clearly,

$$\left| w_{j_1} - \varepsilon\, b^{n-j_1}\,(e^{\frac{2\pi i}{b}l_1} - 1) \right| \leq c.$$

for some constant $c > 0$ since $\varepsilon b^{n-j_1} \leq b$. Hence

$$|\mu_{jml}| \leq \bar{c}\, b^{-n-j_1}.$$

$\square$

Now we are ready to prove the main result.

Proof of Theorem 1.1. Let $\mathcal{R}_n$ be a generalized Hammersley type point set with a_n satisfying $|2a_n - n| \leq c_0$ for some constant $c_0 \geq 0$. Let μ_{jml} be the b-adic Haar coefficients of the discrepancy function of $\mathcal{R}_n$. Theorem 3.1 gave us an equivalent quasi-norm on $S^r_{pq}B(\mathbb{Q}^2)$ so that the proof of the inequality

$$\left(\sum_{j \in \mathbb{N}^2_{-1}} b^{(j_1+j_2)(r-\frac{1}{p}+1)q} \left(\sum_{m \in \mathbb{D}_j,\, l \in \mathbb{B}_j} |\mu_{jml}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq C\, b^{n(r-1)} n^{\frac{1}{q}}$$

for some constant $C > 0$ establishes the proof of the theorem.

We use different parts of Proposition 5.1 after having split the sum by Minkowski's

inequality. We have

$$
\begin{aligned}
&\left(\sum_{j\in\mathbb{N}_0^2;\, j_1+j_2<n-1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\mu_{jml}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\
&\le c_1\left(\sum_{j\in\mathbb{N}_0^2;\, j_1+j_2<n-1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\left(\sum_{m\in\mathbb{D}_j} b^{-2np}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\
&= c_1\left(\sum_{j\in\mathbb{N}_0^2;\, j_1+j_2<n-1} b^{[(j_1+j_2)(r+1)-2n]q}\right)^{\frac{1}{q}}\\
&= c_1\left(\sum_{\lambda=0}^{n-2} b^{[\lambda(r+1)-2n]q}(\lambda+1)\right)^{\frac{1}{q}}\\
&\le c_1 n^{\frac{1}{q}}\left(\sum_{\lambda=0}^{n-2} b^{[\lambda(r+1)-2n]q}\right)^{\frac{1}{q}}\\
&\le c_2 n^{\frac{1}{q}} b^{n(r-1)}
\end{aligned}
$$

from (i). From (ii) we have (using the fact that $\frac{1}{p}-r>0$)

$$
\begin{aligned}
&\left(\sum_{0\le j_1,j_2\le n;\, j_1+j_2\ge n-1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\mu_{jml}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\
&\le c_3\left(\sum_{0\le j_1,j_2\le n;\, j_1+j_2\ge n-1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\, b^{n\frac{q}{p}}\, b^{(-n-j_1-j_2)q}\right)^{\frac{1}{q}}\\
&\quad + c_4\left(\sum_{0\le j_1,j_2\le n;\, j_1+j_2\ge n-1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\, b^{(j_1+j_2)\frac{q}{p}}\, b^{(-2j_1-2j_2)q}\right)^{\frac{1}{q}}\\
&= c_3\left(\sum_{0\le j_1,j_2\le n;\, j_1+j_2\ge n-1} b^{\left[(j_1+j_2)(r-\frac{1}{p})+\frac{n}{p}-n\right]q}\right)^{\frac{1}{q}}\\
&\quad + c_4\left(\sum_{0\le j_1,j_2\le n;\, j_1+j_2\ge n-1} b^{(j_1+j_2)(r-1)q}\right)^{\frac{1}{q}}\\
&= c_3\left(\sum_{\lambda=n-1}^{2n}(2n-\lambda+1)b^{\left[\lambda(r-\frac{1}{p})+\frac{n}{p}-n\right]q}\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&+ c_4 \left(\sum_{\lambda=n-1}^{2n} (2n-\lambda+1) b^{\lambda(r-1)q} \right)^{\frac{1}{q}} \\
&= c_3 b^{\frac{n}{p}-n} \left(\sum_{\lambda=1}^{n+2} \lambda b^{\left[(2n+1-\lambda)(r-\frac{1}{p})\right] q} \right)^{\frac{1}{q}} + c_4 \left(\sum_{\lambda=1}^{n+2} \lambda b^{(2n+1-\lambda)(r-1)q} \right)^{\frac{1}{q}} \\
&\leq c_5 b^{n(r-1)+n(r-\frac{1}{p})} \left(\sum_{\lambda=1}^{n+2} \lambda b^{\lambda(\frac{1}{p}-r)q} \right)^{\frac{1}{q}} + c_6 b^{2n(r-1)} \left(\sum_{\lambda=1}^{n+2} \lambda b^{\lambda(1-r)q} \right)^{\frac{1}{q}} \\
&\leq c_5 b^{n(r-1)+n(r-\frac{1}{p})} (n+2)^{\frac{1}{q}}\, b^{(n+3)(\frac{1}{p}-r)} + c_6\, b^{2n(r-1)} (n+2)^{\frac{1}{q}}\, b^{(n+3)(1-r)} \\
&\leq c_7\, b^{n(r-1)}\, n^{\frac{1}{q}}.
\end{aligned}
$$

Part (iii) gives us (using the fact that $r-1 \leq 0$)

$$
\begin{aligned}
&\left(\sum_{j \in \mathbb{N}_0^2;\, j_1 \geq n} b^{(j_1+j_2)(r-\frac{1}{p}+1)q} \left(\sum_{m \in \mathbb{D}_j,\, l \in \mathbb{B}_j} |\mu_{jml}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\leq c_8 \left(\sum_{j \in \mathbb{N}_0^2;\, j_1 \geq n} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\, b^{(-2j_1-2j_2)q}\, b^{(j_1+j_2)\frac{q}{p}} \right)^{\frac{1}{q}} \\
&= c_8 \left(\sum_{\lambda=n}^{\infty} (\lambda+1) b^{\lambda(r-1)q} \right)^{\frac{1}{q}} \\
&\leq c_9 n^{\frac{1}{q}} b^{n(r-1)}
\end{aligned}
$$

and an analogous result for those $j \in \mathbb{N}_0^2$ with $j_2 \geq n$. From (iv) we conclude

$$
\begin{aligned}
&\left(\sum_{0 \leq j_1 < n;\, j_2 = -1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q} \left(\sum_{m \in \mathbb{D}_j,\, l \in \mathbb{B}_j} |\mu_{jml}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\leq c_{10} \left(\sum_{0 \leq j_1 < n;\, j_2 = -1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\, b^{(j_1+j_2)\frac{q}{p}}\, b^{(-n-j_1)q} \right)^{\frac{1}{q}} \\
&= c_{11} b^{-n} \left(\sum_{j_1=0}^{n-1} b^{j_1 q r} \right)^{\frac{1}{q}} \\
&\leq c_{11} b^{-n} b^{nr} = c_{11} b^{n(r-1)} \leq c_{11} b^{n(r-1)} n^{\frac{1}{q}}.
\end{aligned}
$$

Analogously one estimates the sum for those $j \in \mathbb{N}_{-1}^2$ with $j_1 = -1$ and $0 \leq j_2 < n$.

From (vi) we have

$$\begin{aligned}
&\left(\sum_{n\le j_1;\, j_2=-1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\mu_{jml}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\
&\le c_{12}\left(\sum_{n\le j_1;\, j_2=-1} b^{(j_1+j_2)(r-\frac{1}{p}+1)q}\, b^{(j_1+j_2)\frac{q}{p}}\, b^{-2j_1 q}\right)^{\frac{1}{q}}\\
&= c_{13}\left(\sum_{j_1=n}^{\infty} b^{j_1(r-1)q}\right)^{\frac{1}{q}}\\
&\le c_{13}b^{n(r-1)} \le c_{13}b^{n(r-1)}n^{\frac{1}{q}}
\end{aligned}$$

again with analogous results for the sum with those $j\in\mathbb{N}^2_{-1}$ with $j_1=-1$ and $n\le j_2$. Finally, the last part gives us

$$|\mu_{(-1,-1),(0,0),(1,1)}| \le c_{14}b^{-n} \le c_{14}b^{n(r-1)}n^{\frac{1}{q}}.$$

And the theorem is proved.

□

6 Final remarks

The results from [T10a, Chapter 6] allow us to get additional results for Triebel-Lizorkin spaces with dominating mixed smoothness without any effort. First we define the spaces. We use the notation from the introduction. Let $0<p,q\le\infty$ and $r\in\mathbb{R}$. The Triebel-Lizorkin space with dominating mixed smoothness $S^r_{pq}F(\mathbb{R}^d)$ consists of all $f\in\mathcal{S}'(\mathbb{R}^d)$ with finite quasi-norm

$$\left\|f|S^r_{pq}F(\mathbb{R}^d)\right\| = \left\|\left(\sum_{k\in\mathbb{N}^d_0} 2^{r(k_1+\ldots+k_d)q}|\mathcal{F}^{-1}(\varphi_k\mathcal{F}f)(\cdot)|^q\right)^{\frac{1}{q}}|L_p(\mathbb{R}^d)\right\|$$

with the usual modification if $q=\infty$. The space $S^r_{pq}F(\mathbb{Q}^d)$ can be defined analogously to $S^r_{pq}B(\mathbb{Q}^d)$. In [T10a] we find the following embeddings

$$S^r_{p,\min(p,q)}B(\mathbb{Q}^d) \hookrightarrow S^r_{pq}F(\mathbb{Q}^d) \hookrightarrow S^r_{p,\max(p,q)}B(\mathbb{Q}^d)$$

and

$$S^r_{p_1,q}F(\mathbb{Q}^d) \hookrightarrow S^r_{qq}B(\mathbb{Q}^d) \hookrightarrow S^r_{p_2,q}B(\mathbb{Q}^d)$$

for $0 < p_2 \le q \le p_1 < \infty$. Using the main result of this note and these embeddings we get the following theorem

Theorem 6.1. *Let $1 \le p, q \le \infty$ and $0 \le r < \frac{1}{p}$. Then for any integer $b \ge 2$ there are constants $c_1, c_2 > 0$ such that, for any $N \ge 2$, the discrepancy function of any point set $\mathcal{P}$ in $\mathbb{Q}^d$ with N points satisfies*

$$\left\| D_{\mathcal{P}} | S^r_{pq}F(\mathbb{Q}^d) \right\| \ge c_1\, N^{r-1} (\log N)^{\frac{d-1}{q}},$$

and, for any $n \in \mathbb{N}$ and any generalized Hammersley type point set $\mathcal{R}_n$ with a_n satisfying $|2a_n - n| \le c_0$ for some constant $c_0 > 0$, we have

$$\left\| D_{\mathcal{R}_n} | S^r_{pq}F(\mathbb{Q}^2) \right\| \le c_2\, b^{n(r-1)}\, n^{\frac{1}{q}}.$$

Additionally we get

$$c_1 \frac{(\log N)^{\frac{q(d-1)}{q-1}}}{N^r} \le \mathrm{Err}_N(S^r_{pq}F(\mathbb{Q}^d)) \le c_2 \frac{(\log N)^{\frac{q(d-1)}{q-1}}}{N^r}.$$

The spaces $S^r_pH(\mathbb{Q}^d) := S^r_{p2}F(\mathbb{Q}^d)$ are called Sobolev spaces with dominating mixed smoothness. It is well known that $S^r_pH(\mathbb{Q}^d) = L_p(\mathbb{Q}^d)$. We can conclude the following.

Theorem 6.2. *Let $1 \le p \le \infty$ and $0 \le r < \frac{1}{p}$. Then for any integer $b \ge 2$ there are constants $c_1, c_2 > 0$ such that, for any $N \ge 2$, the discrepancy function of any point set $\mathcal{P}$ in $\mathbb{Q}^d$ with N points satisfies*

$$\left\| D_{\mathcal{P}} | S^r_pH(\mathbb{Q}^d) \right\| \ge c_1\, N^{r-1} (\log N)^{\frac{d-1}{q}},$$

and, for any $n \in \mathbb{N}$ and any generalized Hammersley type point set $\mathcal{R}_n$ with a_n satisfying $|2a_n - n| \le c_0$ for some constant $c_0 > 0$, we have

$$\left\| D_{\mathcal{R}_n} | S^r_pH(\mathbb{Q}^2) \right\| \le c_2\, b^{n(r-1)}\, n^{\frac{1}{q}}.$$

Additionally we get

$$c_1 \frac{(\log N)^{\frac{q(d-1)}{q-1}}}{N^r} \le \mathrm{Err}_N(S^r_pH(\mathbb{Q}^d)) \le c_2 \frac{(\log N)^{\frac{q(d-1)}{q-1}}}{N^r}.$$

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